Development of Cognitive Model in Problem Solving

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Abstract

The aim of this paper is to develop a model of memory for representing the process of students' understanding in solving mathematical problems. The model takes two stages for students to solve mathematical problems. First, students understand what is required to solve a given problem. The understanding of the problem is their goal to problem-solving. Then, they recall a possible procedure to attain the particular goal. The next stage is to determine a working order of scenes under the selected Memory Organization Packet (MOP) and to solve the problem with successive fulfillement of each scene. Our model defines the mechanism necessary to interpret a given problem for selecting the active memory which related to the process of problem solving and has the possibility to or actually represent a thinking process when students solve the problem. It is possible to analyze the several cases of unsuccessful students, who cannot recall any procedure of problem solving and who cannot interpret any given problem. The model was found to be very useful to improve students' academic achievement. As an example, we demonstrated the model of students' thinking processes in the course of solving a geometric problem.

1. Introduction

Nagase, Showji, Iwawaki, and Sakamoto (1988) represent children's thinking process in the course of solving geometric problems on the basis of Schank's Memory Organization Packet (MOP) theory (see Figure 1). The purpose of this paper is to demonstrate a memory structure which is basic to the thinking process, while using a microcomputer as a tool useful for school children in learning and understanding school materials. Employing an information processing approach from the cognitive science, we have analyzed students' memory structure as they solve a given problem.

Using practical and geometrical knowledge which is gained by experience and instruction, most students form an inner representation as a mental model (Johnson-Laird, 1983), when they are given geometric problem. It was necessary for us to capture students' performance in relation to their knowledge as they solved the geometric problem. Students utilize logical knowledge as well as empirical knowledge when facing such a problem.

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2. Memory Structures

2.1 Level model of memory

The present study focused on school children's mental processes while solving a given geometric problem. When school children are given a geometric problem, they first have to understand what the demands of the problem are making reference to the figure and the sentence, and to recognize that their goal is to solve the problem. They make a set of plans for achieving the goal, and select the means to that end. When school children are given the geometric problem which is shown in Figure 2, they understand that their goal is to show that line BE is congruent to line CD while paying attention to line AB and line AC that are on the same side common to each of two similarity, shaped triangles similar to each other. Those who have a knowledge of triangular congruence can make a plan to demonstrate this congruence and then to show the equality of two lines. In the case of Fig. 2, to pay attention to two lines which share one side of two triangles is to understand the meaning of the given problem. Though it is possible...
to try other ways for giving it meaning and to take other plans for solving the problem, we will deal with the ways to discover the meaning for using trianguler congruence, since the subjects are learning to generate the meanings for using congruence in their current math course.

The mechanism which recalls plans and means for solving a given problem on the basis of understanding the meaning of the problem is called Thematic Organization Point (TOP) which Schank (1982) has used as a mechanism to understand the meaning of a story. In the present paper we use the concept “TOP” as a mechanism for understanding the substantial meaning involved in a geometric problem. Knowledge about the particular object (e.g., triangle) provides the basis for thinking during the process of problem-solving. School children have Event Memory (EM) which is a memory for each case and Generalized Event Memory (GEM) a memory for features common to a set of similar cases. Knowledge about geometric features of triangle is included in GEM. TOP retrieves knowledge suitable for solving the geometric problem and then provides plans and means to Intentional Memory (IM). Memory structured to solve the problem is a unit which is called a MOP (Memory Organization Packet). The MOP is composed of the goal and the scenes necessary to attain the goal. The scene is a structured memory corresponding to a section of thinking in the course of problem-solving. The MOP and the scenes in solving the geometric problem shown in Figure 2 are diagramed in Figure 3. Memory in solving the particular problem is called Situational Memory (SM) which contains information about the particular situation (problem) in general.
2.2 Memory about visual experience and memory about mathematics

EM and GEM are the memories for an object and the base for IM and SM. Schank (1980) has discussed EM and GEM on the basis of particular experiences. We classify EM and GEM, which constitute the basis for mental model (Johnson-Laird, 1983), into the "memory about triangles" through visual experiences and the "memory about geometries" which is systematically taught by a teacher.

School children learn mathematical details about triangle (see Table 1) in a geometry class.

Table 1. Some Basic Geometric Vocabulary for triangle

<table>
<thead>
<tr>
<th>Concept</th>
<th>plane, point, line, angle, triangle congruence, true, false</th>
</tr>
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<tbody>
<tr>
<td>Representation</td>
<td>(x,y) coordinates, A(point), $\exists A$(angle)</td>
</tr>
<tr>
<td></td>
<td>$\triangle ABC$(triangle), $ABC \equiv DEF$(congruence)</td>
</tr>
<tr>
<td></td>
<td>&quot;if, hence&quot;</td>
</tr>
</tbody>
</table>
It is important for children to connect new information with known information in learning something. In the case of beginners, knowledge about geometrics is mainly associated with visual memory, including figures which are shown in the textbook and drawn on the blackboard. Most textbooks in Japan state that if one triangle is placed exactly on top of the other, they are congruent. Textbooks describe that the demonstration of a geometric problem is to form a conclusion from any given fact. These are essential to formation and operation of a mental model. For children, triangle and congruence are regarded as teaching materials, and to learn demonstration in mathematics is one goal for them. The first step for them to attain the goal is to solve any given problem with formation and operation of the mental model.

The second step is to learn that knowledge about mathematics is applicable to knowledge from visual experiences. Understanding of the triangle is developed through the mutual relationship between mathematical knowledge and experiential knowledge (see Figure 4). Children store knowledge about the triangle in their memory in such a way that it makes sense to them.

![Figure 4. Mutual relation between mathematical knowledge and experiential knowledge](image)

Children sometimes operate to put one triangle upon another with mental representation, using mental rotation.
Figure 5. Mental rotation

As can be seen in Figure 5, sometimes children mentally place one triangle on top of the other, using mental rotation. Children set up an imaginary center (p) and rotate mentally the triangle ABC around (p). Dotted lines in Figure 5 represent the mental operation which is performed with many elementary procedures corresponding to Johnson-Laird (1983) mental model.

Figure 6. Representation of Mental Model

Figure 6 indicates that triangle ABC and triangle A'B'C' are represented on the mental model and that both triangles relate to each other through mental rotation $R(p)$ which takes the point (p) as a centre. In Figure 2, at first sight triangle ABE may look like being congruent to triangle ACD. To demonstrate the congruence, it is necessary to take procedures with some conditions, such as sharing a part of the sides, keeping a reverse relation, or having the same side. Thus, these triangles in Figure 2 are more complicated than those in Figure 5. If in Figure 2 children guess that triangle ABE may be congruent to triangle ACD, they are able to solve the problem geometrically, performing a series of scenes in Figure 3. This means that vision-based knowledge become coincident with mathematical knowledge. Even if some of children catch visually the congruence, they sometime cannot demonstrate the congruence because they misuse
their mathematical knowledge. In such a case they examine their mental model, congruence conditions, and the relation of MOP to these conditions again and again for finding the final solution.

2.3 Combination of one procedure with another

Mental rotation procedure works with a mental model which is a representation of cognitive structure corresponding to a given problem (the triangle, in this example). MOP which is knowledge for solving the problem is composed of one goal and two or more scenes. Each scene has one subgoal and two or more actions. The MOP theory (Schank, 1982) hypothesizes that action is a minimum unit for information processing. Mental rotation is one of information processing strategies at the action level. Though it is possible to decompose the action into smaller units, it is appropriate for students to have consideration for the unit at their cognizable level in order for it to be helpful for their learning.

Learning is building up scenes to a particular end with combination of solution strategies at the action level, and to build up a MOP with a series of scenes. As the basis for building up the MOP, it is necessary to establish the relation of mathematical knowledge to knowledge from visual images. A goal is needed to combine strategies with each other. The final goal is information obtained from demonstrating the problem. The goal of MOP and the goal of scene are considered as subgoals for the final goal.

3. Cognitive Model for Demonstration of Congruence

3.1 Process of Understanding

The process of understanding is in the whole range of constructing an intentional memory after children read the meaning of a figure (e.g., triangle), when they are given a geometric problem. The main content of the problem given in Figure 2 is as follows:

\[ \text{ABC : isosceles triangle} \]
\[ \text{AD=AE : hypothesis} \]
\[ \text{DC=EB : conclusion} \]

Children get information that the length of DC appears to be identical to EB, based on visual data. They, then, decide that their goal is to demonstrate DC = EB if AD is congruent to AE; that is the goal for intentional memory is DC = EB. As can be seen in Figure 7, children have to represent mathematical meaning for the problem “demonstrate DC = EB.” In Figure 7, MBUILD which is one of primitive elements represents the action to derive “new information from earlier information,” and IDENT-LENGTH represents the relationship that the two lines are congruent.
Any TOP is a collection of information about what happens within the given problem. In the case of Figure 2, the TOP retrieves that information which has a pattern as shown in Figure 7 on the basis of EM and GEM in the MOP. The pattern shown in Figure 7 is found in knowledge about the isosceles triangle (see Figure 8), knowledge about the congruent triangle (see Figure 9) and so on. 
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Figure 9. Knowledge about congruent triangles

Signs in Figure 8 and Figure 9 mean that

IDENT-CONGR: triangle 1 is congruent to triangle 2

*1*; *2*...: point

"MBUILD" fulfills a function which is linked with a procedure that creates new information. The MOP shown in Figure 8 is a MOP for the definition of an isosceles triangle and the MOP in Figure 9 is a MOP for the nature of congruent triangles. Figure 9 is a part of Figure 10, where NONE indicates no relation there, MOP₁ is a MOP for demonstration of congruence and MOP₂ is a MOP for demonstration of geometry.

Figure 10. Knowledge about triangles
3.2 Thinking process to demonstrate the problem

The thinking process of the subjects was modeled when they demonstrated geometric problem. The problem shown in Figure 2 was individually given to 19 children after they learned triangular congruence. The process of demonstration was derived from postprotocols and eye movements. Of 19 children 12 were able to demonstrate the problem, using knowledge about the congruence of triangles. They took steps as follows:

1) understand that this was to demonstrate geometric problem;
2) recall knowledge about congruence of triangle;
3) find two triangles which are visually congruent to each other;
4) examine a side of one triangle corresponding to that of another with angles identical to each other;
5) decide the congruence based on congruent conditions;
6) conclude the demonstration based on a concept of congruence.

Seven of 19 students did not employ knowledge about triangular congruence for solving the problem. Some of them looked for two sides that were congruent to each other and misused knowledge about isosceles triangles, while others looked for two angles congruent to each other.

3.3 Application of Mental Model to Geometric Problem

School children construct a mental model (see Figure 11) which is an inner representation of figures, based on their knowledge about the figure of GEM. Furthermore, they store the procedure for demonstrating the geometric problem in the GEM as a MOP. (see Figure 12). Children try to give a form to the undecided part in Figure 11. In the case of recalling knowledge about congruent relationships across two lines (see Figure 13), such memory is stored together with knowledge about the means ("MOP — use of congruence concept", "MOP — use of definition of isosceles triangle") to show the relationship, if children have already learned the means like that. In the case that Line 1 and 2 in Figure 13 are applied to DC and EB in Figure 2 and are constructed as a mental model, "MOP — use of congruence concept" is recalled with inference from the mental model. If this is connected to the undecided part of "MOP — demonstration of geometric problem" in Figure 12, children can construct their own strategy for demonstration. After this stage children began to solve the given problem, deciding upon the scenes to be used and the order of performance, and then performing these scenes one by one. Those who used "MOP — use of definition of isosceles triangle" could not recall "MOP — use of congruence concept" until their teacher shows them two isosceles triangles as an example. Though they had both "MOP", they recalled only "MOP — use of definition of isosceles triangle" with inference from mental model.

Conclusion

The model we have proposed in the present paper shows you the function of
memory about the goal, the means, and the object and indicates the mechanism to recall the means to the goal and the inference from mental model. The model is found to represent the process of thinking in children's problem-solving in a practical way. Based on the present model, furthermore, it is possible to attempt the analysis of procedures for problem-solving, as well as for the analysis of the difficulty in recalling the means for solving problems or the misuse of some procedures. This suggests that the model can be applied to instruction for slow learners.

![Figure 11. Mental Model](image)

![Figure 12. MOP structure common to demonstration of geometric problem](image)

![Figure 13. Memory about congruent relationship of two lines](image)
References